

Multipole Moments of Static Spacetimes

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In this talk I describe work, mostly done in collaboration with W. Simon some 20 years ago [1,2], on multipole moments of static spacetimes. My purpose is to make this work, which lies at the interface of classical potential theory, conformal geometry and general relativity, known to mathematicians and to perhaps motivate them to have a look at the open problems which still remain.

Our basic object will be an asymptotically flat, smooth or analytic Riemannian 3-manifold $(\widetilde{M}, \widetilde{g}_{ij})$ where \widetilde{M} is diffeomorphic to $\mathbf{R}^3 \setminus \mathbf{B}_{R_0}$ with \mathbf{B}_{R_0} the Euclidean ball of radius R_0 . In the chart given by this diffeomorphism the metric \widetilde{g}_{ij} is required to satisfy the fall-off conditions

$$\begin{aligned} \widetilde{g}_{ij} - \delta_{ij} &= O(1/r), & r^2 &= \delta_{ij} x^i x^j \\ \partial \widetilde{g}_{ij} &= O(1/r^2) \\ \partial^2 \widetilde{g}_{ij} &= O(1/r^3). \end{aligned} \tag{1}$$

It is often convenient and sometimes vital to have a stronger notion of asymptotic flatness which is defined via “conformal compactification” (CC) and goes as follows: There should exist a positive function Ω on \widetilde{M} , such that the metric $g_{ij} = \Omega^2 \widetilde{g}_{ij}$ extends to a smooth, or perhaps analytic metric on $M = \widetilde{M} \cup \{r = \infty\} = \widetilde{M} \cup \{\Lambda\}$, where

$$\Omega|_{\Lambda} = 0, \quad \Omega_i|_{\Lambda} = 0, \quad (\Omega_{ij} - 2g_{ij})|_{\Lambda} = 0, \tag{2}$$

with $\Omega_{i\dots}$ denoting covariant derivatives. There is a gauge freedom associated with this definition. Namely, if Ω gives a CC, then $\overline{\Omega} = \omega\Omega$, with $\omega|_{\Lambda} = 1$, gives another one. Note that, with the above notion of asymptotic flatness, there exists a CC with a rescaled (“unphysical”) metric g_{ij} being C^0 but not C^1 in general. Namely, pick $\Omega = 1/r^2$ and

$$x'^i = x^i / r^2 \quad (\text{“Kelvin inversion”}) \tag{3}$$

as coordinates on M . Then

$$g'_{ij} = (\delta^k_i - n'^k n'_i)(\delta^\ell_j - n'^\ell n'_j) \widetilde{g}_{k\ell} \tag{4}$$

with $n'^i = x'^i / r'^2$, $n'_i = \delta_{ij} n'^j$. Hence

$$\begin{aligned} g'_{ij} &= \delta_{ij} + (\delta^k_i - n'^k n'_i)(\delta^\ell_j - n'^\ell n'_j)(\widetilde{g}_{k\ell} - \delta_{k\ell}) \\ &= \delta_{ij} + O(r'). \end{aligned} \tag{5}$$

Clearly, if \widetilde{g}_{ij} is flat, g_{ij} is also flat, so there is a smooth, even analytic, CC in this case. In some applications where a smooth CC exists, the conformal factor Ω can not be chosen to be smooth. As an important example take the spatial Schwarzschild metric. This is given by

$$\widetilde{g}_{ij} dx^i dx^j = \left(1 + \frac{m}{2r}\right)^4 \delta_{ij} dx^i dx^j \quad (m > 0). \tag{6}$$

Then, taking $x'^i = x^i / r^2$ and $\Omega = r'^2(1 + mr'/2)^2$, we have that $g_{ij} = \delta_{ij}$, but Ω is only C^2 . Clearly, if we had started from δ_{ij} rather than \widetilde{g}_{ij} as the basic metric this problem

would not have arisen. Fortunately, if \tilde{g}_{ij} is the spatial metric of an asymptotically flat static spacetime satisfying the Einstein vacuum equations, as is the Schwarzschild metric, there is a general way of writing $\Omega = f\Omega'$ so that Ω' is smooth. An asymptotically flat vacuum spacetime, for the present purposes, is the manifold $(\mathbf{R} \times \tilde{M}, ds^2)$ with ds^2 given by

$$ds^2 = -e^{2\tilde{U}} dt^2 + \tilde{g}_{ij} dx^i dx^j, \quad (7)$$

with \tilde{U} a function on \tilde{M} satisfying

$$\tilde{U} = O(1/r), \quad \partial\tilde{U} = O(1/r^2), \quad \partial^2\tilde{U} = O(1/r^3). \quad (8)$$

We now rewrite \tilde{g}_{ij} as

$$\tilde{g}_{ij} = e^{-2\tilde{U}} \tilde{\gamma}_{ij}. \quad (9)$$

In the case of the Schwarzschild metric we have that

$$e^{2\tilde{U}} = \left(\frac{1 - m/2r}{1 + m/2r} \right)^4, \quad \tilde{\gamma}_{ij} = \left(1 - \frac{m^2}{4r^2} \right)^2 \delta_{ij}, \quad r > m/2. \quad (10)$$

(Thus R_0 has to be taken $\geq m/2$.) It easily follows that $\tilde{\gamma}_{ij}$ has a smooth CC with smooth conformal factor. The Schwarzschild solution is a solution of the Einstein vacuum equations on $\mathbf{R} \times \tilde{M}$. These equations are equivalent to

$$\Delta_{\tilde{\gamma}} \tilde{U} = 0, \quad (11)$$

$$\mathcal{R}_{ij}[\tilde{\gamma}] = 2\tilde{U}_i \tilde{U}_j, \quad (12)$$

where $\Delta_{\tilde{\gamma}}$ is the Laplace Beltrami operator on $(\tilde{M}, \tilde{\gamma}_{ij})$ and $\mathcal{R}_{ij}[\tilde{\gamma}]$ the Ricci tensor of $\tilde{\gamma}_{ij}$. The equations (11,12) should be viewed as the “source-free static Einstein equations on an exterior domain”. In particular, $(\tilde{U}, \tilde{\gamma}_{ij})$ could have smooth extensions to \mathbf{R}^3 so that ds^2 satisfies the static Einstein equations on $\mathbf{R} \times \mathbf{R}^3$ with an energy momentum tensor corresponding to some “reasonable” matter model. If, for example, this matter model corresponds to a perfect fluid with density ρ , pressure p and an equation of state of the form $p = p(\rho)$ satisfying a certain differential inequality, it is known [3,4] that the spacetime is spherically symmetric (whence Schwarzschild in $\mathbf{R} \times \tilde{M}$), and thus there is just a one-parameter family of solutions, parametrized, say, by the central density. In the trivial case that the spacetime is vacuum throughout $\mathbf{R} \times \mathbf{R}^3$ it is known [5] that ds^2 is the Minkowski metric. (If the matter model describes an elastic solid such as the earth, the solutions will in general be neither Schwarzschild nor flat space, of course.) The point of restricting ourselves to an exterior domain is that we do not care about what sort of matter may be “inside”. Then there is a larger class of solutions. How large, in fact, this class is and how it can be conveniently characterized, is precisely the question we are asking. On physical grounds one expects that there should be the same freedom as in the Newtonian theory, to which we now turn. The Equ.’s (11,12) are now replaced by

$$\Delta_{\tilde{\gamma}} \tilde{U} = 0, \quad \tilde{U} = O(1/r) \quad (13)$$

$$\mathcal{R}_{ij}[\tilde{\gamma}] = 0. \quad (14)$$

Thus, since $\dim \widetilde{M} = 3$, the Riemann tensor of $\tilde{\gamma}_{ij}$ has to vanish. By the simple connect-
edness of \widetilde{M} it follows that $\tilde{\gamma}_{ij} = \delta_{ij}$ in suitable coordinates, and thus there remains

$$\Delta_\delta \tilde{U} = 0, \quad (15)$$

that is to say the Laplace equation with respect to the Euclidean metric. It is convenient
to replace (15) by

$$\Delta_\delta \tilde{U} = -4\pi\rho, \quad (16)$$

where $\rho \in C_0^\infty(\mathbf{R}^3)$, whence \tilde{U} has to be of the form

$$\tilde{U}(x) = \int_{\mathbf{R}^3} \frac{\rho(x')}{|x - x'|} dx'. \quad (17)$$

It follows [6] that \tilde{U} has an expansion of the form

$$\tilde{U} = \frac{M}{r} + \frac{M_i x^i}{r^3} + \frac{M_{ij} x^i x^j}{2r^5} + \dots + \frac{M_{i_1 \dots i_k} x^{i_1} \dots x^{i_k}}{k! r^{2k+1}} + \dots \quad (18)$$

where the constants $M_{i_1 \dots i_k}$ are symmetric and tracefree, which converges uniformly in
 \widetilde{M} (see [6]). As a warm-up for general relativity we outline an independent argument for
this classical result, as follows: One first shows that

$$\tilde{U} = \sum_{\ell=0}^k \frac{M_{i_1 \dots i_\ell} x^{i_1} \dots x^{i_\ell}}{\ell! r^{2\ell+1}} + O(1/r^{k+1}) \quad (19)$$

for some sufficiently large k , say $k = 2$, and that this relation can be differentiated at
least twice (i.e. any partial derivative of the left-hand side is $O(1/r^{k+2})$, a.s.o.). Then
pick the standard CC. Now define an unphysical potential U by $U = \Omega^{-1/2} \tilde{U}$. This has
the form

$$U = M + M_i x'^i + \frac{1}{2} M_{ij} x'^i x'^j + O(r'^3). \quad (20)$$

Thus U is $C^{2,\alpha}$ ($0 < \alpha \leq 1$) in $B_{1/R} = M$. Next observe that U again satisfies the
Laplace equation. This is seen by first recalling the conformal behaviour of the conformal
Laplacian, i.e.

$$\left(\Delta_\gamma - \frac{\mathcal{R}}{8} \right) U = \Omega^{-5/2} \left(\Delta_{\tilde{\gamma}} - \frac{\tilde{\mathcal{R}}}{8} \right) \tilde{U}, \quad U = \Omega^{-1/2} \tilde{U} \quad (21)$$

and specializing to the case where both $\tilde{\gamma}_{ij}$ and γ_{ij} are flat. We now have that U is a
 $C^{2,\alpha}$ -solution of

$$\Delta_\delta U = 0 \quad \text{in } M. \quad (22)$$

But it is well known that solutions to the Laplace equation have to be analytic [7]. Thus
 U has a convergent Taylor expansion at the origin, which ends the proof.

Since analyticity, by [8], is a general property of solutions to elliptic systems with
analytic coefficients, one can try to use a similar line of thought for the relativistic case.

The main obstacle is that the equations (11,12), far from being invariant under the conformal rescaling

$$U = \Omega^{-1/2} \tilde{U}, \quad \gamma_{ij} = \Omega^2 \tilde{\gamma}_{ij}, \quad (23)$$

become formally singular at the point Λ where Ω vanishes. Before showing how this obstacle can be overcome we have to perform the analogue of the first step in the Newtonian case, namely the analysis of the first few terms of the expansion of the quantities $(\tilde{U}, \tilde{\gamma}_{ij})$ in powers of $1/r$. This has been done, up to arbitrary orders in $1/r$, in the sequence of works [9,10,1]. One could obtain sufficiently detailed information about the general term in order to show: For every k there exist coordinates x^i so that, after the standard CC, γ_{ij} is smooth. For example, for $k = 3$, one finds

$$\tilde{\gamma}_{ij} = \delta_{ij} - \frac{M^2}{r^4}(\delta_{ij}r^2 - x_i x_j) - \frac{2MM_{(i}x_{j)})}{r^4} + \frac{2MM_k x^k}{r^6}(-\delta_{ij}r^2 + 2x_i x_j) + O(1/r^4), \quad (24)$$

$$\tilde{U} = \frac{M}{r} + \frac{M_i x^i}{r^3} + \frac{1}{2} \frac{M_{ij} x^i x^j}{r^5} + \frac{M^3}{r^3} + O(1/r^4) \quad (25)$$

for some constants M , M_i , M_{ij} (M_{ij} symmetric and tracefree) and where $x_i = \delta_{ij}x^j$. These equations show in particular that, at least at this order, the relativistic solution has no more free parameters than the Newtonian one, and this remains true for arbitrary high orders.

Very loosely speaking, whereas there are harmonic functions solving Equ. (11) with the appropriate boundary conditions which are parametrized by multipole moments, the metric $\tilde{\gamma}$ is essentially determined by the r.h. side of Equ. (12). This, in turn, is related to the fact that \mathcal{R}_{ij} determines the curvature tensor in 3 dimensions and that zero curvature implies a flat metric. Within the iterative scheme by which the system (11,12) is solved this means that, for each order in $1/r$, “harmonic” contributions to $\tilde{\gamma}_{ij}$ have to be pure gauge.

Armed with this information one can now study a suitable “unphysical” version of Equ.’s (11,12). It is not clear how to use the standard CC for this purpose. The basic observation is that the expansion of \tilde{U} , as in (25), also says that \tilde{U}^2 is arbitrarily smooth in Kelvin inverted coordinates and that, when $M \neq 0$, (Ω, γ_{ij}) satisfies conditions (2) with $\Omega = \tilde{U}^2/M^2$. It is instructive to first see the effect of this CC in the Newtonian case. In that case we have that $\tilde{\mathcal{R}} = 0$ so that $\tilde{\Delta}\tilde{U} = 0$ and $U \equiv M$. Now Equ. (21) implies

$$\mathcal{R} = 0. \quad (26)$$

Conversely we find from $(\tilde{\Delta} - \tilde{\mathcal{R}}/8) \cdot \text{const} = 0$ that, outside Λ , $\Delta(\Omega^{-1}) = 0$. Equivalently

$$\Omega \Delta \Omega = \frac{3}{2} \Omega_i \Omega^i. \quad (27)$$

Of course, the unphysical metric γ_{ij} is now not flat any longer in general. It satisfies

$$-\Omega \mathcal{R}_{ij} = \left(\Omega_{ij} - \frac{1}{3} g_{ij} \Delta \Omega \right). \quad (28)$$

The relation (27), as it stands, does not give rise to a regular elliptic equation for Ω since

$$\sigma := \frac{3}{2} \frac{\Omega_i \Omega^i}{\Omega} \quad (29)$$

is formally singular at Λ . Using (2.8) we find that

$$\sigma_i = -3\mathcal{R}_{ij}\Omega^j \quad (30)$$

which further implies

$$\Delta\sigma = 3\Omega\mathcal{R}_{ij}\mathcal{R}^{ij}. \quad (31)$$

Now (31), together with $\Delta\Omega = \sigma$, *does* form a regular elliptic system. Using the theorem of Morrey [8], it would now follow that (Ω, σ) are analytic provided γ_{ij} is. This, in turn, follows from the following differential geometric

Lemma: A conformally flat metric with zero scalar curvature is analytic.

Proof: By $\mathcal{R} = 0$ and conformal flatness there holds

$$D_k \mathcal{R}_{ij} = D_i \mathcal{R}_{kj}. \quad (32)$$

Taking D^k of Equ. (32), commuting derivatives and using

$$\mathcal{R}_{kji\ell} = 2\mathcal{R}_{i[k}g_{j]\ell} - 2\mathcal{R}_{\ell[k}g_{j]i} \quad (33)$$

and the Bianchi identity we obtain

$$\Delta\mathcal{R}_{ij} = 3\mathcal{R}_i^k \mathcal{R}_{jk} + g_{ij} \mathcal{R}_{k\ell} \mathcal{R}^{k\ell}. \quad (34)$$

Now recalling that the Ricci tensor of a metric g_{ij} , when written in harmonic coordinates, gives an elliptic operator for γ_{ij} , we see that

$$\mathcal{R}_{ij} = \sigma_{ij}, \quad (35)$$

together with (34) yields an elliptic system for $(\gamma_{ij}, \sigma_{ij})$ with analytic coefficients. Thus, by [8], the Lemma is proved.

In the case at hand we have that \mathcal{R} vanishes and, furthermore, that γ_{ij} is conformally flat. In fact, from (29) we can directly obtain

$$\mathcal{R}_{i[j;k]} = 0. \quad (36)$$

Combining the analyticity of γ_{ij} with the system (30,31) we obtain analyticity of the pair (Ω, γ_{ij}) .

The relativistic situation, given by Equ.'s (13,14), is similar, but more complicated. Instead of $\Omega = \tilde{U}^2/M^2$ it is more convenient to take

$$\Omega = \frac{4}{M^2} \sinh^2 \frac{\tilde{U}}{2} \quad (37)$$

as conformal factor. As a field variable it is useful to take, instead of Ω , the quantity ρ , defined by

$$\rho = \tanh^2 \frac{\tilde{U}}{2}. \quad (37')$$

Then, again, it turns out that $\mathcal{R} = 0$ and

$$\Delta\rho = \sigma := \frac{3}{2} \frac{\rho_i \rho^i}{\rho}, \quad (38)$$

whereas (28) gets replaced by

$$-\rho(1-\rho)\mathcal{R}_{ij} = \rho_{ij} - \frac{1}{3}\gamma_{ij}\Delta\rho. \quad (39)$$

Clearly the metric γ_{ij} can not any longer be conformally flat. In fact, taking the “curl” of (39), Equ. (36) of the Newtonian theory gets replaced by

$$(1-\rho)\mathcal{R}_{i[j;k]} = 2\mathcal{R}_{i[j}\rho_{k]} + \gamma_{i[j}\mathcal{R}_{k]}\ell\rho^\ell. \quad (40)$$

As a side remark we add, that, from the above equation it is not hard to show that γ_{ij} is conformally flat iff (Ω, γ_{ij}) comes from the Schwarzschild solution.

For σ we obtain, instead of (31), the relation

$$\Delta\sigma = 3\rho(1-\rho)^2 + 3\mathcal{R}_{ij}\rho^i\rho^j. \quad (41)$$

Apropos Equ. (41) we wish to point out that, with the substitution

$$\kappa := \frac{\sigma^{1/4}}{(1-\rho)^{1/2}} \quad (42)$$

and using (40), there results the identity

$$\Delta\kappa = \frac{1}{2}\kappa^5 + \frac{9}{16}\kappa^{-7}\mathcal{R}_{i[j;k]}\mathcal{R}^{i[j;k]}, \quad (43)$$

which plays an important role in black hole uniqueness theory (see e.g. [11]).

In any case, we can now consider the system consisting of (35), (38), (41) and the equation of the form

$$\Delta\sigma_{ij} = \dots \quad (44)$$

obtained by multiplying (40) by $(1-\rho)^{-1}$ (which is non-zero near Λ) and taking D^k . This furnishes an elliptic system with analytic coefficients for the variables $(\rho, \gamma_{ij}, \sigma, \sigma_{ij})$ (that-is-to-say if harmonic coordinates are used in Equ. (35)). We conclude that (Ω, γ_{ij}) is analytic.

In the Newtonian case, the multipole moments of the solution were nothing but the Taylor coefficients at Λ of the unphysical potential U in the standard CC. In particular this implies that these moments determine the physical solution uniquely (which however

already follows from the expansion (18)). In the relativistic case, and using the CC above based on $\Omega = \tilde{U}^2/M^2$, it is not immediately clear whether the moments determine the solution. In fact, the expansions in (24,25) use a specific coordinate condition. In the unphysical picture one would like a “gauge invariant” definition of multipole moments. Luckily, Geroch in [12] came up with such a definition. It goes as follows: Consider the following recursively defined set of tensor fields built from (U, Ω, γ_{ij})

$$\begin{aligned} P &= U \\ P_i &= U_i \\ P_{ij} &= U_{ij} - \frac{1}{3}\gamma_{ij}\Delta U \\ P_{i_1\dots i_{s+1}} &= \mathcal{C} \left[D_{i_{s+1}} P_{i_1\dots i_s} - \frac{s(2s-1)}{2} \mathcal{R}_{i_1 i_2} P_{i_3\dots i_{s+1}} \right], \end{aligned} \quad (45)$$

where \mathcal{C} denotes the operation of taking the tracefree symmetric part. Now define

$$M_{i_1\dots i_s} := P_{i_1\dots i_s}|_{\Lambda}. \quad (46)$$

Clearly, in the Newtonian case and using the standard CC for which $\mathcal{R}_{ij} = 0$, this definition coincides with the standard one. As for the gauge dependence, when

$$\bar{\Omega} = \omega\Omega, \quad \bar{U} = \omega^{-1/2}U, \quad \bar{\gamma}_{ij} = \omega^2\gamma_{ij}, \quad (47)$$

with $\omega|_{\Lambda} = 1$ it turns out [13] that

$$\bar{M}_{i_1\dots i_s} = M_{i_1\dots i_s} + \mathcal{C} \sum_{r=0}^{s-1} \binom{s}{r} (2s-1) \dots (2r+1) (-1)^{r-s} M_{i_1\dots i_s} b_{i_{r+1}} \dots b_{i_s}, \quad (48)$$

where $b_i := \frac{1}{2}\omega_i|_{\Lambda}$. Note that only first derivatives of ω at Λ enter the transformation law (48). Furthermore the dependence of $M_{i_1\dots i_s}$ on b_i is exactly the same as the change of the Newtonian moments under a translation of the physical Euclidean space by the vector b_i . We can now come back to the conformal gauge with $\Omega = \tilde{U}^2/M^2$. Here it follows from (45) that $U \equiv M$ so that the moments are equivalent to M , together with \mathcal{C} of the derivatives of \mathcal{R}_{ij} at Λ . But it is not difficult to infer from the above equations that these data determine (Ω, γ_{ij}) uniquely whence $(\tilde{U}, \tilde{\gamma}_{ij})$ is also determined. We remark, finally, that the $M_{i_1\dots i_s}$ -terms in the physical expansion, which we have written out for $k=2$ in (24,25), coincide with the Geroch moments from above.

There are two problems which the above analysis leaves open. The first one is to give an analyticity proof in the case where M is zero. Although the “physical” expansion of $(\tilde{U}, \tilde{\gamma}_{ij})$ seems to be completely insensitive to whether M is nonzero or not, the unphysical situation, because of the choice of conformal factor makes vital use of the nonvanishing of M . The second open issue is that of convergence. Namely, given a set of moments $M_{i_1\dots i_s}$, $s = 0, 1, \dots$: what is the condition on the behaviour for large s so that a static vacuum solution having these as multipole moments exists? The Newtonian situation is

clear: the moments have to be such that the Taylor series having them as coefficients converges. In the relativistic case, the Taylor coefficients at Λ of the quantities (Ω, γ_{ij}) , say in Riemannian normal coordinates, depend on these moments in a nonlinear fashion. In particular it is not even obvious whether a solution exists for which only finitely many moments are nonzero.

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